Fluids in Flatland - A Short Introduction

~ apologies to Edwin Abbott

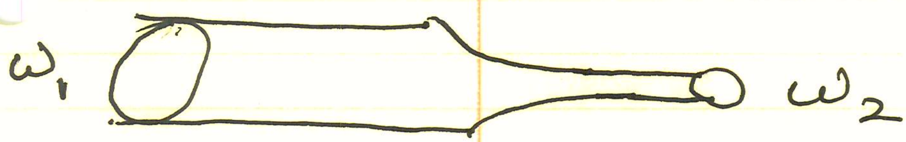
→ A Quick Look: (i.e. Why Notable)
(Sneak preview)

- what physical process underpins K41 cascade, etc. ?
  → vortex tube stretching

i.e. $\partial_t \underline{\omega} = \underline{\nabla} \times (\underline{v} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}$

$$\partial_t \underline{\omega} + \underline{v} \cdot \underline{\nabla}\, \underline{\omega} - \nu \nabla^2 \underline{\omega} = \underline{\omega} \cdot \underline{\nabla}\, \underline{v}$$

$\nabla \underline{v} \longrightarrow$



- 2D $\quad \underline{\omega} \cdot \underline{\nabla}\, \underline{v} = 0 \qquad (\underline{\nabla} \cdot \underline{v} = 0)$

so $\quad \partial_t \underline{\omega} + \underline{v} \cdot \underline{\nabla}\, \underline{\omega} = \nu \nabla^2 \underline{\omega} + \underline{f}_{ext}$

$$\partial_t \nabla^2 \phi + \underline{\nabla} \phi \times \hat{z} \cdot \nabla \nabla^2 \phi = \nu \nabla^2 \nabla^2 \phi + \tilde{f}_{ext}$$

$\therefore$ two inviscid quadratic invariants:

$\int \frac{v^2}{2} \longrightarrow$ energy

$\int \frac{\omega^2}{2} \longrightarrow$ enstrophy (~ mean squared vorticity)

<u>Refs</u>:

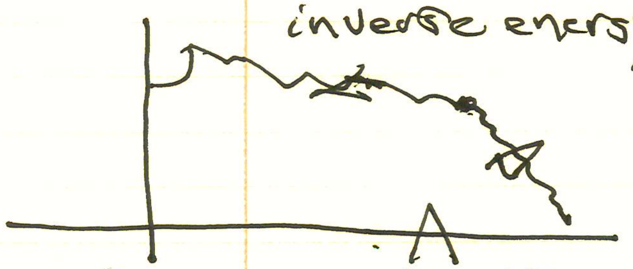
- R. Salmon : Notes on GFD
- G. Vallis : Atmospheric and Oceanic Fluids

+ Posted Notes from GFD Module
+ Posted References (both Module & Lectures)

N. B. Boffetta and Ecke review especially recommended.

Key element in 2D turbulence is constraint imposed on dynamics by dual conservation law.

Upshot: Dual Cascade (Kraichnan '67)



dual self-similarity ranges.

$\Rightarrow$ and drag $\partial_t \omega + \underline{V}\cdot\underline{\nabla}\omega + \mu\omega = \nu\nabla^2\omega + f$

<u>Why 2D ?</u> $\rightarrow$ Constrained Dynamics.

- Recall Taylor - Proudman Theorem

$\rightarrow$ in rotating fluid, $(\underline{\omega} + 2\underline{\Omega})/\rho$ is 'frozen in'.

$\rightarrow$ $\Omega \gg$ other rates

$2\Omega\,\partial_z \underline{V} \cong 0$

$\Rightarrow$ ~2D dynamics

Immediately realize that

~2D dynamics $\Leftrightarrow$

characteristic of ~2D dynamics

$Ro = V/L\Omega < 1$

Rossby #

– (V/L) → 'other rates'

i.e. $\underline{v}\cdot\underline{\nabla}\,\underline{v}$ vs $2\underline{\Omega}\times\underline{v}$

– contrast Re

⇒ favors slow, large scale motion in (thin) rotating system

i.e. atmosphere, ocean, etc.

Ways to 2D-ize:

- rotation , $Ro = V/L\Omega$
- stable stratification

  $\# \sim V/LN$ $\quad N^2 = \frac{g}{\rho_0}\frac{\partial\rho_0}{\partial z}$

- strong magnetic field

  $Ro \to V/L\Omega_{ci}$ ⇒ Hasegawa–Mima model

  cyclotron frequency

– Low Ro dynamics

Given Ro << 1, have fundamental relation between pressure and vorticity

includes centrifugal

$$\frac{d\underline{v}}{dt} = -\underline{\nabla}\left(p^*/\rho\right) - 2\underline{\Omega}\times\underline{v}$$

Ro << 1 ⇒ Geostrophic balance

$$0 \simeq -\nabla(p/\rho) - 2\underline{\Omega}\times\underline{v}$$

$$\Rightarrow \quad \underline{v}_\perp = \Omega\times\nabla(p^*/\rho)/\Omega^2$$

$$\boxed{\underline{v}_\perp = -\frac{\underline{\nabla}(p^*/\rho)\times\hat{z}}{\Omega}}$$

$p^*/\rho \leftrightarrow \phi$

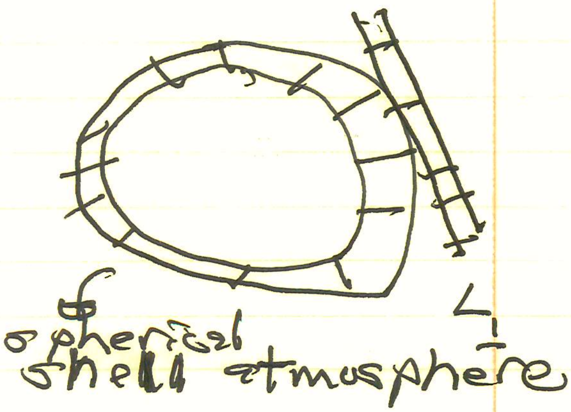
Pressure-as-stream-function:

clockwise
counter-clockwise

Fluid rotation about Low / High pressure cells.

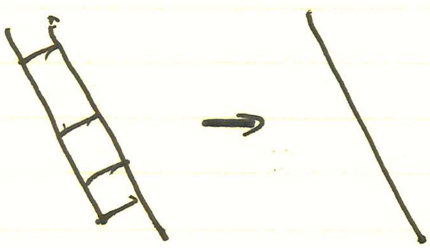
— β-plane Model

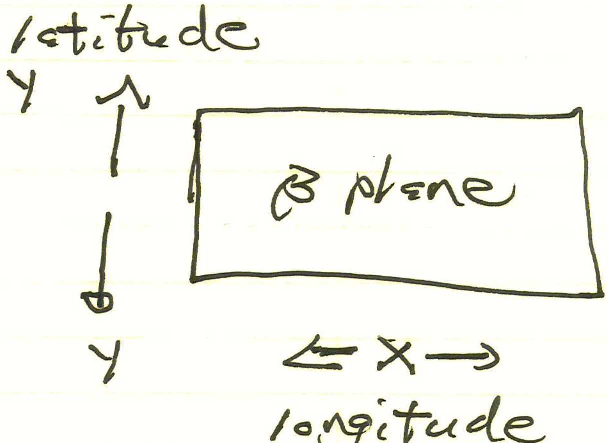
→ Quick derivation



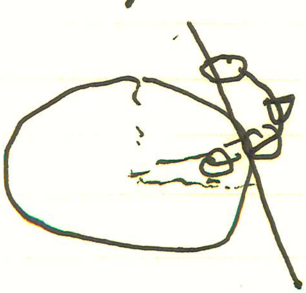
- β plane tangent to spherical shell atmosphere
- strong stable stratification on $L_\perp$.

so describe/approximate dynamics in 2D plane tangent to sphere i.e. β-plane





Now, consider displacement of fluid/vortex element:



- $\underline{\omega} + 2\underline{\Omega}$ frozen in
- $C = \int d\underline{a} \cdot (\underline{\omega} + 2\underline{\Omega})$

circulation conserved.

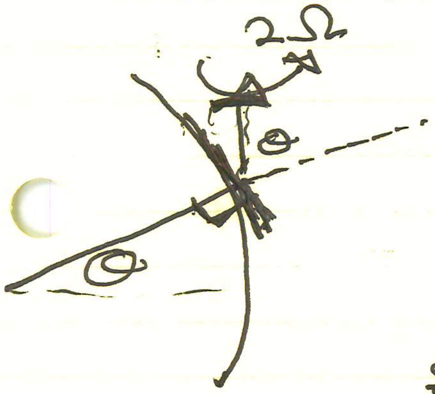
Point: displacing fluid element implies change in

$$\int d\underline{s} \cdot \underline{\Omega} \sim \hat{n} \cdot \hat{z} \sim \cos \Theta_p$$

$\Downarrow$ polar K

$\therefore$ there must be a change in <u>fluid vorticity</u> to conserve circulation

since planetary vorticity piece of circulation changed by displacement



$A \equiv$ area of vortex

$$\frac{dG}{dt} = 0$$

$$\frac{d}{dt}\left(A\omega + A 2\Omega \sin\Theta\right) = 0$$

$\Downarrow$ projection factor

$\Rightarrow$

$$\frac{d\omega}{dt} = -2\Omega \cos\Theta \frac{d\Theta}{dt}$$

$$= -\frac{2\Omega}{R} \cos\Theta \frac{d}{dt}(R\Theta)$$

$$= -\beta V_y$$

$$\beta = \frac{2\Omega \cos\Theta}{R}$$

$\rightarrow$ Θ gradient in Coriolis force.

Of course $\frac{d}{dt}(R\theta) = \frac{d}{dt} y = v_y$

$$\boxed{d\omega/dt = -\beta v_y}$$

+ add dissipation, forcing

Charney

$$\partial_t \omega + \underline{v} \cdot \underline{\nabla} \omega + \mu \omega = \nu \nabla^2 \omega + f$$

$$d/dt = \partial_t + \underline{v} \cdot \underline{\nabla}$$

$\hat{z} \perp \beta$ plane

$$\underline{v} = -\frac{\underline{\nabla}\phi}{2\Omega} \times \hat{z} \rightarrow \nabla \phi \times \hat{z}$$

$$\omega = \nabla_\perp^2 \phi / 2\Omega$$

$R \rightarrow \infty$, $\beta \rightarrow 0$

$$\boxed{\begin{aligned} \partial_t \nabla^2 \phi + \nabla \phi \times \hat{z} \cdot \nabla \nabla^2 \phi + \mu \nabla^2 \phi \\ = \nu \nabla^2 \nabla^2 \phi + f \end{aligned}}$$

2D fluid equation.

→ simplest incarnation of "2D Fluid"
⟺ 2D turbulence (eddies)

→ β-plane equation is next simplest ⟹ supports waves, eddys, zonal flows.

Observe :

can re-write 2D inviscid equation as

$$\partial_t \omega_z + \{\omega_z, H\} = 0$$

$H = \phi$ Conservative Hamiltonian evolution

similar to Liouville or Vlasov equation:

$$\partial_t f + \{f, H\} = 0$$

$$H = \frac{p^2}{2m} + |e|\phi \quad , \quad + \text{ Poisson's equation}$$

$$\partial_t f + v\,\partial_x f + \frac{Z}{M} E\,\partial_v f = 0$$

i.e. $\omega_z \Leftrightarrow f \Rightarrow$ $\left\{ \begin{array}{l} \text{conserved (phase} \\ \text{space) density.} \end{array} \right.$

which brings us to:

$\boxed{\text{Potential Vorticity}}$

Observe can write equations in conservative form, i.e.

$$\frac{d}{dt}\omega = 0 \qquad \text{(pure 2D)}$$

$$\frac{d}{dt}(\omega + \beta y) = 0 \qquad (\beta\text{-plane})$$

fluid vorticity (ω)

βy → planetary vorticity (l.o. in expansion)

$\omega + \beta y \equiv$ simple example of potential vorticity (PV)

- generalized vorticity, akin to phase space density

"GFD = the study of Fluids with PV"

= "The Fluid Dynamics of PV"

More generally on PV:

- recall for rotating fluid:

$$\frac{d}{dt}\left(\frac{\underline{\omega} + 2\underline{\Omega}}{\rho}\right) = \frac{(\underline{\omega} + 2\underline{\Omega})\cdot\underline{\nabla}\underline{V}}{\rho}$$

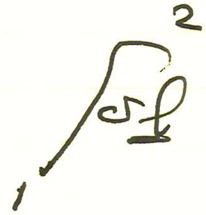
akin.

$$\frac{d}{dt}\delta\underline{l} = \delta\underline{l}\cdot\underline{\nabla}V$$

same eqn → $\dfrac{\underline{\omega} + 2\underline{\Omega}}{\rho}$ frozen in.

Now, consider conserved scalar field ψ

$$\frac{d}{dt}\psi = 0$$



$$\frac{d}{dt}(\psi_1 - \psi_2) = 0$$

$$\delta\psi = \underline{\nabla}\psi \cdot \delta\underline{\ell}$$

so

$$\frac{d}{dt}\left(\underline{\nabla}\psi \cdot \delta\underline{\ell}\right) = 0$$

and $\delta\ell \leftrightarrow \dfrac{\underline{\omega} + 2\underline{\Omega}}{\rho}$

so if $\delta\underline{\ell}$ satisfies, $\dfrac{\underline{\omega} + 2\underline{\Omega}}{\rho}$ must satisfy

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{(\underline{\omega} + 2\underline{\Omega})\cdot\underline{\nabla}\psi}{\rho}\right) = 0}$$

along trajectories

$\hookrightarrow$ general statement of PV conservation

$$\boxed{q = \frac{\underline{\omega} + 2\underline{\Omega} \cdot \underline{\nabla}\psi}{\rho}}$$

PV (general) any $\underline{\nabla}\psi$

i.e.

– H-M: $\rho = \rho_0 + \tilde{\rho}$

$\hat{n} = (\mathrm{e}|\tilde{\phi}/T) n_0$

$\nabla \psi = \hat{z}$

– PV conservation $\leftrightarrow$ particle re-labeling symmetry

(i.e. particles can be re-labeled without changing thermodynamic state)

N.B. If consider finite thickness shell

$$q = \nabla_\perp^2 \phi + \beta y + \underbrace{\frac{f_0^2}{\rho} \partial_z \left(\frac{\rho}{N^2} \partial_z \phi \right)}$$

$f_0 = 2\Omega \sin\theta$ – rotation

$N^2 = g/L_\rho$ – buoyancy

Relevance of finite thickness?

Scale! $\rightarrow$ $1/L_\perp^2$ vs $\dfrac{f_0^2}{N^2 H^2}$

H $\rightarrow$ layer thickness

$\Rightarrow$ (deformation radius)$^{-2}$

$\sim 1/L_d^2$

So

$1/L^2 \sim 1/L_d^2$ → relative vorticity and deformation effects contribute equally

$\left\{ \begin{array}{l} \sim 100\,km \text{ ocean} \\ \sim 1000\,km \text{ atmosphere} \end{array} \right.$

$L < L_d$ → β-plane.

→ 2D Turbulence

- issues: conservation of energy, enstrophy
- trends in constrained spectral evolution
- self-similarity ranges, inverse cascade
- fate of energy

Issues:

- 2D turbulence is *the* generic problem of GFD
- $\beta \to 0$

$$\partial_t \nabla^2 \phi + \underline{\nabla}\phi \times \hat{z} \cdot \underline{\nabla} \nabla^2 \phi - \nu \nabla_\perp^2 \nabla_\perp^2 \phi + \mu \nabla_\perp^2 \phi = \tilde{f}$$

$\tilde{f}$ ↓ any scale

μ ↓ drag → scale invariant damping → control large scale

2 inviscid invariants:

$\langle (\underline{\nabla}\phi)^2 \rangle \to$ energy

$\langle (\nabla^2 \phi)^2 \rangle \to$ enstrophy

N.B.:

- in 3D, enstrophy produced:

$$\frac{d}{dt}\langle \omega^2\rangle \sim \langle \underline{\omega}\cdot(\underline{\omega}\cdot\nabla\underline{v})\rangle$$

$$\langle \Omega(k)\rangle \sim k^2 k^{-5/3} \sim k^{1/3}$$

- 2D, $\underline{\omega}\cdot\underline{\nabla}\,\underline{v} \rightarrow 0$

  → all powers $\int d^2x\, \omega^n$ conserved

  $\int d^2x\, \omega^2 \rightarrow \langle \omega^2\rangle$ conserved on finite box

∴ story incompatible with K41

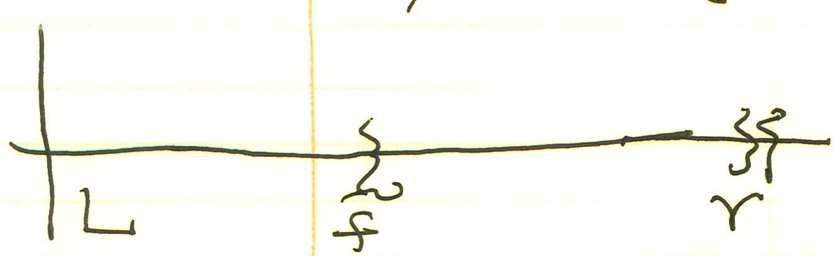
Problem of 2D Fluid:

- given forcing at any scale l_f s/t

  $L \geq l_f > l_v$

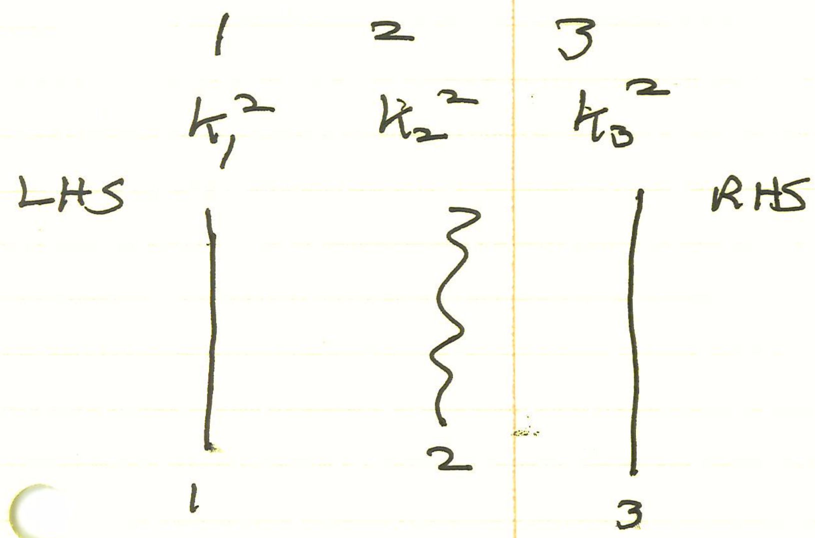
  → how does dual conservation of E, Ω constrain transfer?

  → self-similarity ranges?



Theoretical "clues":

- consider 3 modes, interacting (3 to conserve quadratic quantity)



$k_1^2 < k_2^2 < k_3^2$

$k_2^2 \leftrightarrow k_p^2$

Conservation:

$$E_2 = E_1 + E_3$$

$$\Omega_2 = \Omega_1 + \Omega_3$$

but $\Omega(k) = k^2 E(k)$

$$\begin{cases} E_2 = E_1 + E_3 \\ k_2^2 E_2 = k_1^2 E_1 + k_3^2 E_3 \end{cases}$$

$$\therefore \quad E_1 = \left(\frac{k_3^2 - k_2^2}{k_3^2 - k_1^2}\right) E_2 \rightarrow E_1 \sim E_2$$

$$E_3 = \left(\frac{k_2^2 - k_1^2}{k_3^2 - k_1^2}\right) E_2 \rightarrow E_3 \sim \frac{k_2^2}{k_3^2} E_2$$

as $\Omega(k) = k^2 E(k)$

$E_1 \sim E_2 \Rightarrow$ energy transferred to large scale mode!?

$\Omega_3 \sim \Omega_2 \rightarrow$ enstrophy transferred to small scale mode!?

~ suggests: energy accumulates at large scale, enstrophy accumulates at small scale.

$\Rightarrow$ $\underline{\underline{2}}$ self-similar transfer ranges in 2D turbulence!?

N.B. Analogy (restricted): Asymmetric Top.

Conserve: $\sum_i L_i^2 = L^2$

$E = \sum_i L_i^2 / 2I_i$

$$L^2 = L_1^2 + L_2^2 + L_3^2$$

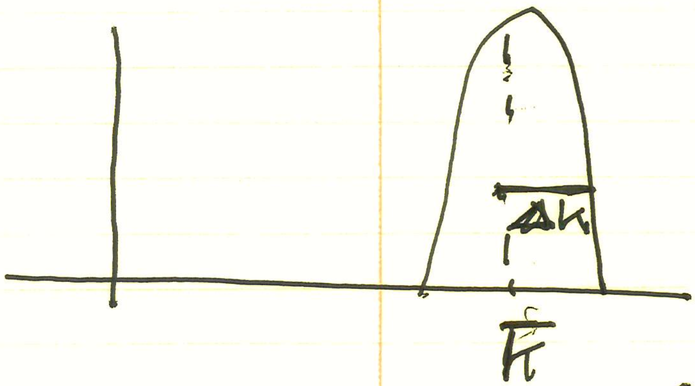
$$E = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}$$

$I \Leftrightarrow 1/k^2$, etc.

and energizing intermediate axis $\rightarrow$ decay to 1, 3 etc.

→ But many D.O.F.'s ...

(Rhines; after the fact)



Consider a spectral 'slug' of turbulence, initialized.

How will $\bar{k}$ evolve, given $\partial_t \langle (\Delta k)^2 \rangle > 0$?
i.e. assume spectrum spreads ...

N.B. Does $\langle \Delta k^2 \rangle$ exist?

$$\langle (\Delta k)^2 \rangle = \int dk\, (k-\bar{k})^2\, E(k) \Big/ \int dk\, E(k)$$

$$= \int dk\, (k^2 - 2k\bar{k} + \bar{k}^2)\, E(k) \Big/ \int dk\, E(k)$$

$$= \int dk\, \left(k^2 E(k) - 2k\bar{k}E(k) + \bar{k}^2 E(k) \right) \Big/ \int dk\, E(k)$$

$$= \left(\Omega_0 - 2\bar{k}^2 E_0 + \bar{k}^2 E_0 \right) / E_0$$

$$= \Omega_0 / E_0 - \bar{k}^2$$

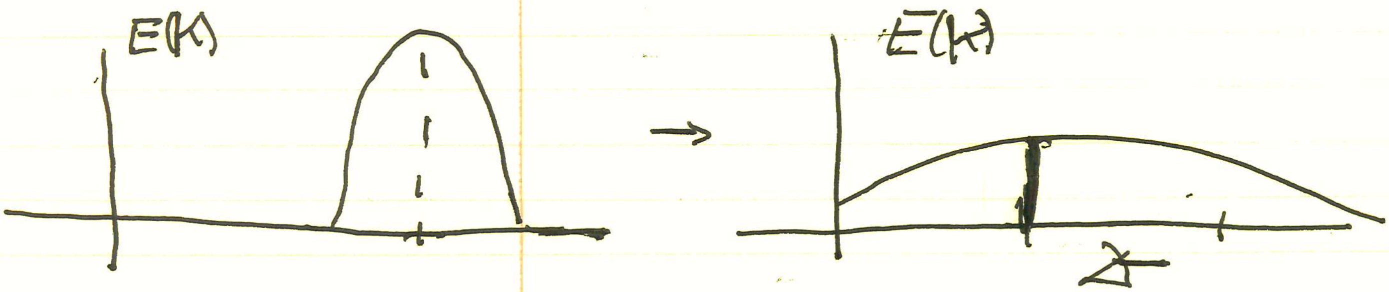
$$= (\Omega_0 / E_0) - \bar{k}^2$$

Here: $\int dk\, E(k) = E_0 \rightarrow$ const

$\int dk\, k^2 E(k) = \Omega_0 \rightarrow$ const

$\int dk\, k\, E(k) = \bar{k}\, E_0 \rightarrow$ defines centroid

so $\partial_t \langle \Delta k^2 \rangle > 0 \Rightarrow \partial_t \bar{k} < 0$



- spectrum broadens but also shifts toward large scales
- energy content shuffled / coupled to larger scale

$\Rightarrow$ suggestive of inverse energy cascade

$\rightarrow$ similar story for enstrophy $\Rightarrow$ forward cascade!

$\therefore$ Enter the Dual Cascade!

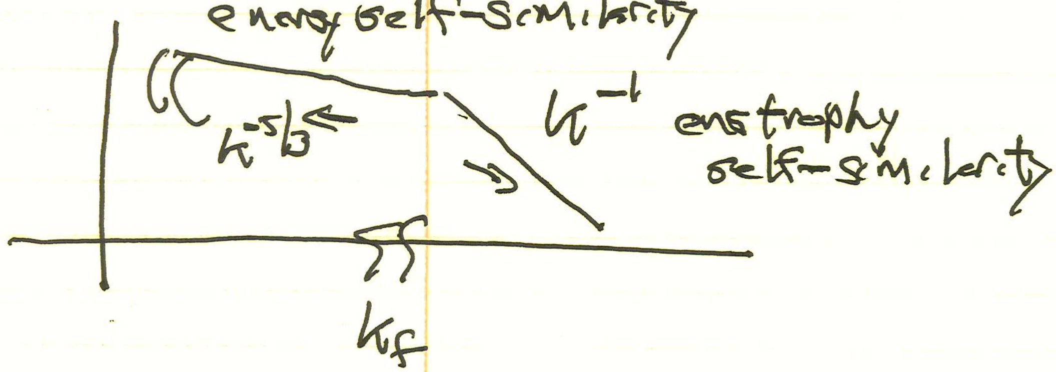
Dual cascade (Kraichnan '67) :

from forcing, system supports $\underline{\underline{2}}$ self-similarity ranges :

- forward enstrophy range / cascade ($k > k_f$)
  - → no forward energy flux
  - → no energy dissipation by viscosity ($Re \to \infty$)

- <u>inverse</u> energy cascade
  - → no inverse energy flux
  - → no viscous power dissipation
  - → damping by drag, etc.
  - → not stationary

Cascade ≡ range self-similar transfer



$$\eta = \frac{d}{dt}\langle \omega^2 \rangle \sim \left(v_f / l_f \right)^3$$

$$\epsilon = \frac{d}{dt}\langle v^2 \rangle \sim v_f^3 / l_f \quad \to$$ forcing rate, not dissipation

of course $k_f^2 \epsilon \sim \eta$.

- Forward ~~→~~ Enstrophy

$$\frac{\Omega(\ell)}{T(\ell)} \sim \eta \qquad \frac{v(\ell)}{\ell} \sim T(\ell)^{-1}$$

$$\sim \omega(\ell)$$

$$\omega(\ell)^3 \sim \eta \qquad \Omega(\ell) \sim \omega(\ell)^2$$

$$\omega^2(\ell) \sim \eta^{2/3}$$

but $\omega^2(\ell) \sim k\,\Omega(k)$

$$\Rightarrow \boxed{\begin{array}{l} \Omega(k) \sim \eta^{2/3}/k \\ \underbrace{E(k)}_{\text{energy spectrum in enstrophy range}} \sim \Omega(k)/k^2 \sim \eta^{2/3}/k^3 \end{array}}$$

- no forward energy flux in enstrophy range
- observe $1/\tau_\ell \sim \eta^{1/3} \to$ const, here

vs K41 $1/\tau_\ell \sim \frac{\epsilon^{1/3}}{\ell^{2/3}} \to$ faster for smaller

⇒ tip-off that since all scales transfer at same rate, non-local transfer of enstrophy can occur

⇒ Corrections? [Logarithmic due straining]

— Inverse Energy

$\epsilon = v(\ell)^2 \dfrac{v(\ell)}{\ell}$, as before but $\epsilon \equiv$ energy stirring rate

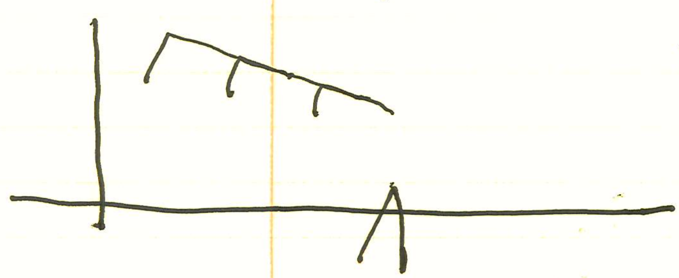
⇒

$E(k) = \epsilon^{2/3} k^{-5/3}$ ⇒ inverse energy cascade range

→ akin 3D, but upscale

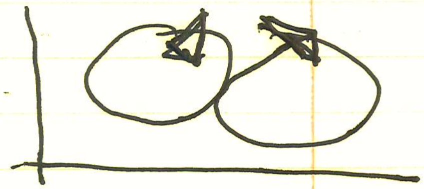
→ $1/\tau(\ell) \sim \dfrac{v(\ell)}{\ell}$ → cascade slows as larger scales approached.

→ not stationary state



largest scale slowest, keeps evolving

→ eventually encounters drag, boundary, etc.



→ straining (non-local) effects in scale ?

→ structure of large scale.

→ no inverse enstrophy flux in energy range.

→ no forward energy flux, P_d by viscosity → 0.
⇓
dissipated power

→ so, where does the energy go ?
- friction μ
- boundary effects
- straining on small scales (Boffetta, et al. posted)

→ $\langle \delta u_L(l)^3 \rangle = +\frac{3}{2} \epsilon l$

~ analogue of 4/5 for inverse energy range

~ + for $l > l_f$

~ ϵ here is stirring rate.

→ (static) vorticity contours in inverse energy range exhibit statistics of percolation cluster !

→ no "conventional" intermittency in inverse cascade. Deviations from Gaussian occur

→ what of particles, particle dispersion?

Revisiting Richardson:

$l_{1,2}$ → energy range

$$\frac{d\,l_{1,2}}{dt} = V(l_{1,2}) = \epsilon^{1/3} l^{1/3}$$

$$l_{1,2}^2 \sim \epsilon\, t^3 \quad ; \quad \text{as before}$$

$l_{1,2}$ → enstrophy

$$\frac{dl}{dt} = v(l) = [\Omega(l)]^{1/2} l = \eta^{1/3} l$$

⇒ separation grows exponentially in enstrophy range

N.B. : Dual cascade used to justify selective decay – minimum enstrophy

$$\Omega = \int d^2x\,(\nabla^2\phi)^2 + \lambda \int d^2x\,(\nabla\phi)^2$$

$$\delta\Omega = 0.$$

Additional Material → Jets.

→ β-Plane : Turbulence, Waves, Flows

Recall:

$$\partial_t \nabla^2\phi + \nabla\phi\times\hat{z}\cdot\nabla\nabla^2\phi - \nu\nabla^2\nabla^2\phi + \mu\nabla^2\phi = -\beta V_y + \tilde{f}$$

Ignoring: ν, μ, $\tilde{f}$

$$\partial_t \nabla^2\phi + \nabla\phi\times\hat{z}\cdot\nabla\nabla^2\phi = -\beta V_y$$

⇒ Waves

$$\omega_k = -\beta k_x / k^2 , \qquad V_{g_y} = \frac{2\beta k_x k_y}{(k^2)^2}$$

→ Rossby wave

and

⇒ flows

How does large scale order emerge?

$$\left.\begin{array}{l} k_x \to 0 \\ k_y \text{ finite} \\ \omega_k \to 0 \end{array}\right\}$$

Zonal Mode

jets, belts, jet stream

2 new players → waves, flows.

Numerous questions:

② → how } do zonal flows form?
why }
⇒ many ways!

① → how do {waves / flows} modify, interact with inverse cascade?

③ → scale of zonal flows?

④ → implications for atmospheric phenomenology

On zonal flows:

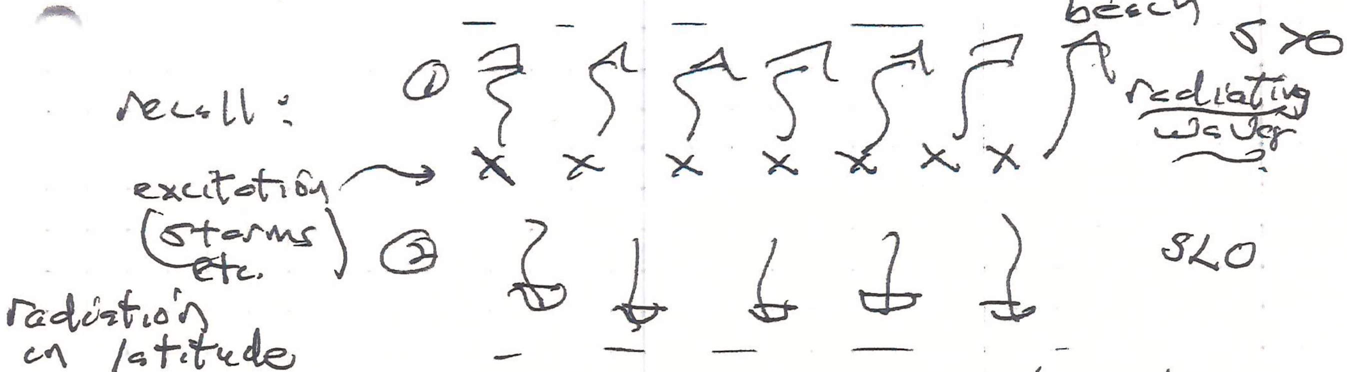
Reynolds stress

↑

- ZFs ubiquitous
- Flows produced by momentum transport
- simplest perspective ⇔ wave propagation!

(c)

(Linear) wave propagation can account for ZF formation!

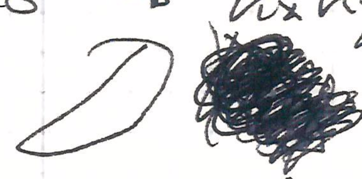


$$\underline{S} = v_{gy}\,\mathcal{E}\,\hat{y} = \frac{2k_x k_y \beta \mathcal{E}}{(k^2)^2}\,\hat{y}$$

beach (absorber) (useful cartoon)

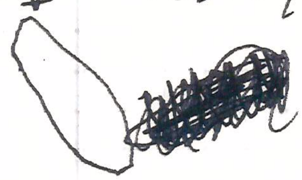
① outgoing waves → $k_x k_y > 0$

$S \sim (+)\,\hat{y}$



$(\omega > 0)$

② $S \sim -\hat{y}$ → $k_x k_y < 0$



$(\omega > 0)$

eddy tilt changes

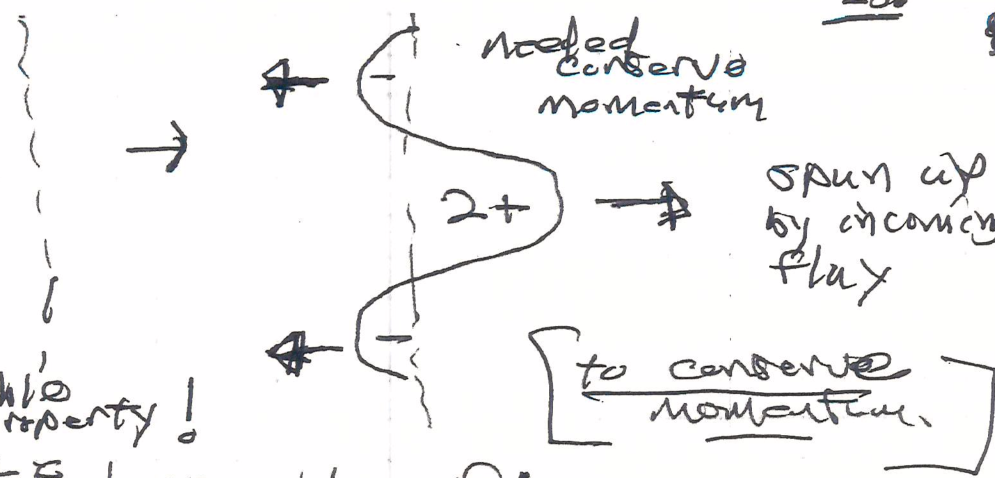
but

$$\langle \tilde{V}_y \tilde{V}_x \rangle = \sum_{k} -k_x k_y |\phi_k|^2$$

so ① → $\Pi_{y,x} < 0$

② → $\Pi_{y,x} > 0$

point: outgoing wave energy density flux generates incoming momentum fluxes



Remarkable property!

→ beautiful example of:

... "the central result that a rapidly rotating flow, when stirred in a localized region, will converge angular momentum into this region" Isaac Held ('01)

(stirring → spin-up!)

Flows ⟺ energy stirring

→ Wave mechanism requires separation of excitation and dissipation (beach) regions.

→ Requires:

- waves
- vorticity / momentum transport in space
- * irreversibility → outgoing waves
- symmetry breaking, ∇ has direction
- sep. forcing / damping

⇒ Useful to investigate wave theorems for flow production → {something general} Why?

→ key observation: PV flux
(inhomogeneous PV mixing)

$$q = \beta y + \nabla^2\phi$$

recall essence of PV conservation forces planetary – flow vorticity exchange.

$\langle \tilde{v}_y \tilde{q} \rangle_z$ → zonal avg.

$$= \langle \tilde{v}_y \nabla^2\phi \rangle_z$$

$$= \langle (-\partial_x\phi)(\partial_x^2\phi + \partial_y^2\phi) \rangle_x$$

but: $\langle \partial_x\phi \, \partial_x^2\phi \rangle = \left\langle \partial_x \left[\frac{(\partial_x\phi)^2}{2} \right] \right\rangle_x = 0$

symmetry!

$$\langle \tilde{v}_y \tilde{q} \rangle_z = -\langle (\partial_x\phi)(\partial_y^2\phi) \rangle$$

$$= -\partial_y \langle \partial_x\phi \, \partial_y\phi \rangle_x + \langle \cancel{\partial_{xy}^2\phi} \, \partial_y\phi \rangle$$

$$\langle \partial_x \frac{(\partial_y\phi)^2}{2} \rangle_x \to 0.$$

$$= \partial_y \langle \tilde{v}_y \tilde{v}_x \rangle_y$$

Taylor Identity

$$\boxed{\langle \tilde{v}_y \tilde{q} \rangle_z = \partial_y \langle \tilde{v}_y \tilde{v}_x \rangle_z}$$

(contrast 3D) – EP.

z dropped hereafter.

→ Reynolds force drives flow!

⇒ Look at potential enstrophy balance.

⇒ Zonally averaged latitudinal PV flux = zonally averaged latitudinal Reynolds force ⇒ drives flow.

As Reynolds stress controls flow: i.e.

$$\rho\left(\frac{\partial V_x}{\partial t} + \underline{V}\cdot\underline{\nabla}V_x\right) = \cancel{-\partial_x P} - \cancel{(2\underline{\Omega}\times\underline{V})_x}$$

cancel ⇒ geostrophic balance

so

$$\boxed{\frac{\partial\langle V_x\rangle}{\partial t} = -\partial_y\langle\tilde{V}_y\tilde{V}_x\rangle + \nu\partial_y^2\langle V_x\rangle - \mu\langle V_x\rangle}$$

then $\left.\begin{array}{l}\text{PV evolution}\\ \text{Potential Enstrophy}\end{array}\right\}$ necessarily control flow.

⇒ What are essential to ZF generation:

- inhomogeneous PV mixing / transport in space
- translation symmetry in direction of the flow.

Now, consider P.E balance: $\underline{3/n}$ (forcing)

$$\frac{d}{dt} q - \nu \nabla^2 q = 0$$

$$\frac{\partial}{\partial t}\tilde{q} + \underline{v}\cdot\nabla\tilde{q} - \nu\nabla^2\tilde{q} = -\tilde{v}_y \frac{d\langle q\rangle}{dy}$$

potential enstrophy evolution

so

$$\frac{\partial}{\partial t}\left\langle \frac{\tilde{q}^2}{2}\right\rangle + \partial_y \underbrace{\left\langle \tilde{v}_y \frac{\tilde{q}^2}{2}\right\rangle}_{\text{Flux of potential enstrophy}} + \underbrace{\nu\langle(\nabla\tilde{q})^2\rangle}_{D_\Omega \rightarrow \text{ dissipation}}$$

$$= \underbrace{-\langle \tilde{v}_y \tilde{q}\rangle \frac{d\langle q\rangle}{dy}}_{\text{potential enstrophy production (flux - gradient)}}$$

$$\left(\frac{d\langle q\rangle}{dy}\right)^{-1}\left[\partial_t\left\langle\frac{\tilde{q}^2}{2}\right\rangle + \partial_y\left\langle\tilde{v}_y\frac{\tilde{q}^2}{2}\right\rangle + \nu\langle(\nabla\tilde{q})^2\rangle\right]$$

$$= -\langle \tilde{v}_y \tilde{q}\rangle = -\langle \tilde{v}_y \nabla^2\tilde{\phi}\rangle$$

but Mean (zonal) flow

$$\partial_t\langle v_x\rangle = -\frac{\partial}{\partial y}\langle \tilde{v}_y\tilde{v}_x\rangle - \mu\langle v_x\rangle$$

$$= -\langle \tilde{v}_y \nabla^2\tilde{\phi}\rangle - \mu\langle v_x\rangle$$

$$-\langle \tilde{v}_y \nabla^2 \tilde{\phi} \rangle = -\left(\partial_t \langle V_x \rangle + \mu \langle V_x \rangle\right)$$

$\therefore$

WAD

$$\partial_t \left\{ \langle V_x \rangle + \frac{\langle \tilde{\xi}^2 \rangle}{2\, d\langle \xi \rangle / dy} \right\} = -\nu \frac{\langle (\nabla \tilde{\xi})^2 \rangle}{d\langle \xi \rangle / dy} - \partial_y \frac{\langle \tilde{v}_y \tilde{\xi}^2 / 2 \rangle}{d\langle \xi \rangle / dy} - \mu \langle V_x \rangle$$

Wave Activity Density

WAD

pseudomomentum to

$$\partial_t \left\{ \langle V_x \rangle - \left(\frac{-k_x \langle \tilde{\xi}^2 \rangle}{2 k_x\, d\langle \xi \rangle / dy} \right) \right\}$$

$$= -\mu \langle V_x \rangle - \delta \langle \tilde{\xi}^2 \rangle / \frac{d\langle \xi \rangle}{dy}$$

absent { – drag, – damping, – mixing (3^{rd} order) } $\Rightarrow$ { Flow ~~locked~~ to wave momentum density } (Charney – Drazin Thm.)

non-acceleration thm!

ZF's $\Leftrightarrow$ wave momentum density

~~Cannot~~ accelerate (or maintain vs drag) zonal flow without changing (decreasing) wave intensity.

Note:
$$\frac{-k_x\langle \tilde{q}^2\rangle}{2\,k_x\, d\langle q\rangle/dy}$$

$$q = \nabla^2\phi + \beta y$$

absent mean flow,

$$\frac{d\langle q\rangle}{dy} = \beta \;!$$

$$\frac{\langle \tilde{q}^2\rangle}{2} = k^2 \mathcal{E}$$

$$\frac{-k_x k^2 \mathcal{E}}{k_x \frac{d\langle q\rangle}{dy}} = \frac{k_x \mathcal{E}}{\frac{-k_x \Omega}{k^2}} = \frac{k_x \mathcal{E}}{\omega_k}$$

$\longrightarrow$ Action Density

$$= k_x\, N_k$$

$$= P_w.$$

i.e. a/a/ Adiabatic Invariant

$\Rightarrow$ wave momentum density

$$\partial_t \left\{ \langle V_x\rangle - P_w \right\} = -\mu\langle V_x\rangle - \delta\langle \tilde{q}^2\rangle / \frac{\partial\langle q\rangle}{\partial y} + \cdots$$

$$u \sim \frac{v(\ell)}{\ell} \sim \epsilon^{1/3} \ell^{-2/3}$$

$$\ell^{2/3} \sim \cancel{u/\epsilon^{1/3}} \quad \epsilon^{1/3}/u$$

$$\ell \sim \epsilon^{1/2}/u^{3/2} < L$$